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Existence and Partial Characterization of the Global Attractor for the Sunflower Equation

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This paper is devoted to the study of some global dynamical properties of the so-called sunflower equation $\varepsilon x''(t) + ax'(t) + b\sin x(t-\varepsilon) = 0$, $\varepsilon > 0$. It is shown that considering this equation as retarded differential equation on $S^1 \pm \mathbb{R}$, it has a global attractor for any $\varepsilon > 0$, which is homeomorphic to S^1 for any $\varepsilon \in [0, a/b]$.

1. INTRODUCTION

The purpose of this work is to study global dynamical properties of the so-called sunflower equation

$$\varepsilon x''(t) + ax'(t) + b \sin x(t-\varepsilon) = 0, \qquad \varepsilon > 0, \tag{1.1}$$

where $x(t)$ is the angle of the plant with the vertical, the time lag ε is the geotropic reaction, and a and b are positive parameters which can be obtained experimentally. For more details, we refer the reader to [5]. Rewriting (1.1) as a system and considering it in the space $C = C([-\varepsilon, 0], \mathbb{R}^2)$, we see that it cannot have a global attractor in C, i.e., a maximal compact invariant set that attracts all bounded sets of C. This is due to the fact that the global attractor must contain all constant solutions $(k\pi, 0)$, $k \in \mathbf{Z}$. In order to avoid this difficulty, we shall consider (1.1) as a retarded functional differential equation (RFDE) on the cylinder $S^1 \times \mathbb{R}$, and we prove that (1.1) as a RFDE on $S^1 \times \mathbb{R}$ has a global attractor S_ε for

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any $\varepsilon > 0$. We also give a partial characterization of the global attractor. More precisely, we prove that S_ε is homeomorphic to S^1 for all $\varepsilon \in (0, a/b]$. In [4] the author proved that $S_\varepsilon \to S_0$ as $\varepsilon \to 0^+$, where S_0 is a circle in $C^\circ = C([-1, 0], S^1 \times \mathbb{R})$, i.e., small delay does not matter for the sunflower equation.

This work is organized as follows. In Section 2, we prove the existence of the global attractor of the sunflower equation. In Section 3, we characterize the global attractor for $\varepsilon \in (0, a/b]$.

2. Existence of a Global Attractor

Let $C = C([-1, 0], \mathbb{R}^2)$ with the usual sup-norm. Rescaling Eq. (1.1) with $\tau = t/\varepsilon$, $\dot{()} = d/d\tau$, $u(\tau) = x(\varepsilon\tau)$, it becomes equivalent to the following two-dimensional system:

$$\begin{aligned} \dot{u}(\tau) &= v(\tau) \\ \dot{v}(\tau) &= -av(\tau) - b\varepsilon \sin u(\tau - 1). \end{aligned} \tag{2.1}$$

We consider (2.1) for $\tau > 0$ subject to the initial conditions

$$u(\tau) = \phi, \quad \tau \in [-1, 0]; \qquad v(0) = v_0, \tag{2.2}$$

where $\phi \in C([-1, 0], \mathbb{R})$ and $v_0 \in \mathbb{R}$.

From [3, p. 18], it follows that the system (2.1) defines a second order RFDE on $S^1 \times \mathbb{R}$ given by the map

$$F : C^\circ \to T(S^1 \times \mathbb{R}), \tag{2.3}$$

where

$$F(\psi_1, \psi_2) = (\psi_1(0), \psi_2(0); \psi_2(0), \qquad -a\psi_2(0) - b\varepsilon \sin \psi_1(-1) u_{\psi_1(0)}),$$

and $u_{\psi_1(0)}$-unit vector tangent to S^1. Since F is globally Lipschitz, we get the existence for any $\tau \geq 0$, uniqueness and the continuity of the solutions of the initial data.

Let us define the solution map by $T_\varepsilon(\tau)\varphi = y_\tau(\varphi, \varepsilon)$, where $y_\tau(\varphi, \varepsilon)$ is the unique solution of F with initial condition $y_0 = \varphi$; and

$$y(\varphi, \varepsilon)(\tau) = (\cos u(\tau, \varphi), \sin u(\tau, \varphi), v(\tau, \varphi)).$$

From [3, thm. 2.2, p. 11], it follows that for each $\varepsilon > 0$ the family $\{T_\varepsilon(\tau)\}_{\tau \geq 0}$ is a strongly continuous semigroup of operators on C°.

Let $M = S^1 \times \mathbb{R}$ be the submanifold of $\mathbf{R}^3$ endowed with a Riemannian structure induced by $\mathbb{R}^3$ with δ_M the associated complete metric. Let

$$\delta(\varphi, \psi) = \max_{\theta \in [-1,0]} \delta_M(\varphi(\theta), \psi(\theta)).$$

be the induced admissible metric on C° by δ_M.

The following result shall be very useful in proving the point dissipativeness of (2.3).

LEMMA 2.1. *Let ρ be the Euclidean metric in $\mathbb{R}^3$. Then*

$$\rho(P, Q) \leq \delta_M(P, Q) \leq \pi\rho(P, Q), \qquad \forall P, Q \in S^1 \times \mathbb{R}.$$

Proof. From the Fig. 1, it is easy to obtain that

$$\sin\theta = \frac{\rho(p, q)}{2}.$$

Taking into account that in this case δ_{S^1} is the arc-length of S^1, we have that $\delta_{S^1}(p, q) = 2\theta$. Combining the last two relations we obtain:

$$\delta_{S^1}(p, q) = \frac{\theta}{\sin\theta}\rho(p, q).$$

This immediately implies:

$$\rho(p, q) \leq \delta_{S^1}(p, q) \leq \frac{\pi}{2}\rho(p, q), \qquad \forall (p, q) \in S^1. \tag{2.4}$$

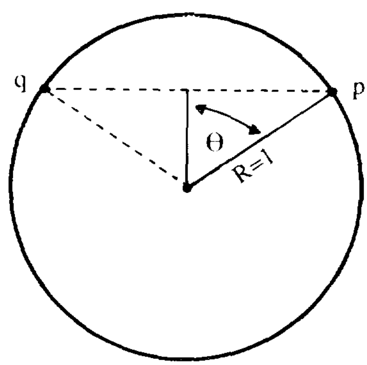

FIGURE 1

Suppose now that $P, Q \in M = S^1 \times \mathbb{R}$. It is obvious that

$$\rho(P, Q) \leq \delta_M(P, Q). \tag{2.5}$$

On other hand, we have (see Fig. 2)

$$\delta_M(P, Q) \leq \delta_M(P, Q^*) + \delta_M(Q^*, Q), \tag{2.6}$$

Since P and Q^* belong to S^1, from (2.4) we get

$$\delta_M(P, Q^*) \leq \frac{\pi}{2}\rho(P, Q^*). \tag{2.7}$$

From Fig. 2, we obtain that:

$$\delta_M(Q^*, Q) = \rho(Q^*, Q). \tag{2.8}$$

Thus, (2.5)–(2.8), imply:

$$\rho(P, Q) \leq \delta_M(P, Q) \leq \frac{\pi}{2}\rho(P, Q^*) + \rho(Q^*, Q) \leq \left(\frac{\pi}{2} + 1\right)\rho(P, Q),$$
$$\forall P, Q \in S^1 \times \mathbb{R}.$$

This proves our claim. ∎

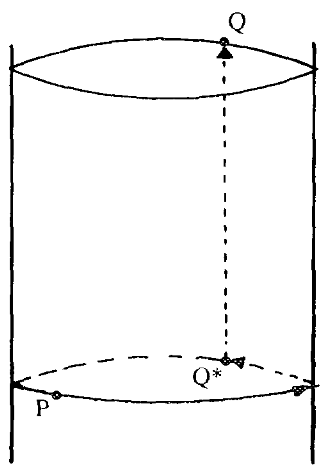

FIGURE 2

LEMMA 2.2. *For each $\varepsilon > 0$, the RFDE on $S^1 \times \mathbb{R}$ given by (2.3) is point dissipative, and the solution map $T_\varepsilon(\tau) : C^\circ \to C^\circ$ is relatively compact for any $\tau \geq 1$.*

Proof. From the second equation of the system (2.1), we obtain

$$v(\tau) = v_0 e^{-a\tau} \ \varepsilon b \int_0^\tau e^{-a(\tau-s)} \sin u(s-1)\, ds.$$

or

$$|v_\tau| \leq |v_0| e^{-a(\tau-1)} + \frac{\varepsilon b}{a}, \qquad \tau \geq 0. \tag{2.9}$$

Let $\varphi^*(\theta) = (1, 0, 0)$, $\theta \in [-1, 0]$. It is clear that $\varphi^* \in C^\circ$, and $y_\tau(\varphi^*)(\theta) = (1, 0, 0)$, $\forall \theta \in [-1, 0]$ and $\tau \in \mathbb{R}$.

Let us set

$$B_\varepsilon = \left\{\varphi \in C^\circ : \delta(\varphi, \varphi^*) \leq 2\pi \left(1 + \left(\frac{\varepsilon b}{a}\right)^2\right)^{1/2}\right\}. \tag{2.10}$$

From Lemma 2.1 we have that

$$\delta_M(P, Q) \leq \pi\rho(P, Q), \qquad \forall P, Q \in S^1 \times \mathbb{R}.$$

Hence,

$$\begin{aligned}\delta_M(y_\tau(\varphi)(\theta), y_\tau(\varphi^*)(\theta)) &\leq \pi\rho(y_\tau(\varphi)(\theta), y_\tau(\varphi^*)(\theta)) \\ &\leq \pi\sqrt{4 + (v(\tau+\theta))^2},\end{aligned} \tag{2.11}$$

for all $\theta \in [-1, 0]$, $\tau \geq 0$.

From (2.9) and (2.11) it is not difficult to show that

$$\delta_M^2(y_\tau(\varphi)(\theta), y_\tau(\varphi^*)(\theta)) \leq 4\pi^2 \left(1 + \left(\frac{\varepsilon b}{a}\right)^2\right) + 2\pi^2 v_0^2 e^{-2a(\tau-1)}. \tag{2.12}$$

Thus, the semigroup $\{T_\varepsilon(\tau)\}_{\tau \geq 0}$ is point dissipative with respect to the set B_ε given by (2.10).

Let us prove now that the solution map $T_\varepsilon(\tau) : C^\circ \to C^\circ$ is relatively compact for any $\tau \geq 1$. Actually, the proof is an application of the Arzela–Ascoli Lemma. Let B be a bounded subset of C°. In order to prove that $T_\varepsilon(\tau)B$ is relatively compact for $\tau \geq 1$, it is sufficient to show that

$$\dot{y}(\varphi, \varepsilon)(\tau) = (-\dot{u}(\tau, \varphi) \sin u(\tau, \varphi), \dot{u}(\tau, \varphi) \cos u(\tau, \varphi), \dot{v}(\tau, \varphi))$$

is bounded for $\tau \geq 0$, and $\varphi \in B$. Taking into account that

$$|\dot{u}(\tau)| = |v(\tau)|$$
$$|\dot{v}(\tau)| \leq a \mid v(\tau) \mid + b\varepsilon,$$

and the inequality (2.9), we obtain that $\dot{y}(\varphi, \varepsilon)(\tau)$ is bounded for $\tau \geq 0$ and $\varphi \in B$. ▮

COROLLARY 2.3. *The following assertions hold*:

(i) *Orbits of bounded sets are bounded.*

(ii) *The semigroup* $\{T_\varepsilon(\tau)\}_{\tau\geq 0}$ *is completely continuous for* $\tau \geq 1$. *In particular*, $\{T_\varepsilon(\tau)\}_{\tau\geq 0}$ *is asymptotically smooth.*

Proof. Let $B \subset C^\circ$ be an arbitrary bounded set. Thus, there exists an $R > 0$ such that $\delta_M(\varphi(\theta), \varphi^*(\theta)) \leq R$ for all $\varphi \in B$ and $\theta \in [-1, 0]$, where $\varphi^* = (1, 0, 0)$. From this relation we get that the third component of $\varphi(0)$ satisfies the inequality $|v_0| \leq R$. Then the first assertion of the corollary, it is an immediate consequence of the inequality (2.12).

By Lemma 2.2 the semigroup $\{T_\varepsilon(\tau)\}_{\tau\geq 0}$ is relatively compact for any $\tau \geq 1$. Thus, $\{T_\varepsilon(\tau)\}_{\tau\geq 0}$ is conditionally completely continuous for $\tau \geq 1$. Since orbits of bounded sets are bounded, we obtain that $\{T_\varepsilon(\tau)\}_{\tau\geq 0}$ is completely continuous for $\tau \geq 1$. In particular, [2, Cor. 3.2.2] implies that $\{T_\varepsilon(\tau)\}_{\tau\geq 0}$ is asymptotically smooth. ▮

PROPOSITION 2.4. *For each* $\varepsilon \geq 0$, *the RFDE on* $S^1 \times \mathbb{R}$ *given by* (2.3) *has a connected global attractor* S_ε, *which is upper semicontinuous in* ε.

Proof. The existence of the global attractor follows directly from [2, Thm. 3.4.6], Lemma 2.2, and Corollary 2.3 above. Since the set B_ε given by (2.10) is connected and $\psi(B_\varepsilon) = S_\varepsilon$, it follows from [2, Lem. 3.1.1] that the global attractor is connected as well.

Finally, fix $\varepsilon^* \geq 0$ and consider $\varepsilon_1 \in (0, \varepsilon^*]$. The set S_{ε_1} attracts the closed ball B_ε^*. Since the semigroup $\{T_\varepsilon(\tau)\}_{\tau\geq 0}$ depends continuously on ε, this is enough to show that S_ε is upper semicontinuous at ε_1. ▮

3. CHARACTERIZATION OF THE GLOBAL ATTRACTOR FOR $\varepsilon \in (0, a/b]$.

The next question of interest is to determine the flow on the global attractor S_ε. We shall give a partial characterization of S_ε showing that the system (2.3) is gradient for $0 < \varepsilon \leq a/b$; i.e.,

(a) Each bounded positive orbit is precompact. In our case, this follows from Corollary 2.3 and [2, Lem. 3.2.1].

(b) There exists a Liapunov function $V : C^\circ \to \mathbb{R}$ such that:

(b_1) $V(\varphi)$ is bounded below,

(b_2) $V(\varphi) \to \infty$, as $\varphi \to \infty$,

(b_3) $V(T_\varepsilon(\tau)\varphi)$ is nonincreasing in τ for each φ in C°,

(b_4) If φ is such that $T_\varepsilon(\tau)\varphi$ is defined for $\tau \in \mathbb{R}$ and $V(T_\varepsilon(\tau)\varphi) = V(\varphi)$ for $\tau \in \mathbb{R}$, then φ is an equilibrium point.

The existence of the Liapunov functional was shown in [1, p. 125]. The functional $V : C^\circ \to \mathbb{R}$ defined by

$$V(\psi_1, \psi_2) = \frac{1}{2}\psi_2^2(0) + \varepsilon b(1 - \cos \psi_1(0) u_{\psi_1(0)}) + \frac{a}{2}\int_{-1}^{0}\int_{s}^{0} \psi_2^2(u)\, du\, ds,$$

where $u_{\psi_1(0)}$-unit vector tangent to S^1, is bounded below and $V(\psi_1, \psi_2) \to \infty$, as $\varphi = (\psi_1, \psi_2) \to \infty$. Differentiating V along the solutions of (2.1), it follows $\dot{V}(T_\varepsilon(\tau)\varphi) \leq 0$, $\forall \tau \geq 0$, and $0 < \varepsilon \leq a/b$. Hence, $V(T_\varepsilon(\tau)\varphi)$ is nonincreasing in τ for each $\varphi \in C^\circ$ and $\varepsilon \in (0, a/b]$. Finally, if φ is such that $T_\varepsilon(\tau)\varphi$ is defined for all $\tau \in \mathbb{R}$ and $V(T_\varepsilon(\tau)\varphi) = V(\varphi)$ for $\tau \in \mathbb{R}$, then φ is an equilibrium point, i.e., $\varphi = (1, 0, 0)$ or $\varphi = (-1, 0, 0)$. *Statements above imply that* $\{T_\varepsilon(\tau)\}_{\tau \geq 0}$ *is a gradient system for any* $\varepsilon \in (0, a/b]$; *and* [2, *Thm.* 3.8.5] *implies that* $S_\varepsilon = W^u(E)$, *if each element of* $E = \{\varphi^* = (1, 0, 0), \varphi^{**} = (-1, 0, 0)\}$ *is hyperbolic*. Thus, a detail local analysis of the system (2.1) will be important in the characterization of the global attractor of the sunflower equation.

To this end, let us consider now the linear systems

$$\begin{aligned} \dot{u}(\tau) &= v(\tau) \\ \dot{v}(\tau) &= -av(\tau) - b\varepsilon u(\tau - 1) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \dot{u}(\tau) &= v(\tau) \\ \dot{v}(\tau) &= -av(\tau) + b\varepsilon u(\tau - 1) \end{aligned} \tag{3.2}$$

which are the linearization of (2.1) around the equilibria $(0, 0)$, $(\pi, 0)$ respectively. Let us consider as well the characteristic equations

$$\Delta_\varepsilon(\lambda) = \lambda^2 + a\lambda + \varepsilon b e^{-\lambda} = 0, \tag{3.3}$$

$$\Delta_\varepsilon^*(z) = z^2 + az + \varepsilon b e^{-z} = 0, \tag{3.4}$$

corresponding to (3.1) and (3.2), respectively. The following result holds:

PROPOSITION 3.1. (i) *There exists a constant* ε_0, $a < \varepsilon_0 b < a\pi/2$, *such that*:

(a) *For any* $\varepsilon < \varepsilon_0$ *all solutions of* (3.3) *have negative real parts.*

(b) *For* $\varepsilon = \varepsilon_0$ *there is a conjugate pair of pure imaginary solutions.*

(c) *For each* $\varepsilon > \varepsilon_0$ *there are precisely two roots of* (3.3) *with* $\mathbb{R}e\lambda > 0$ *and* $-\pi/\varepsilon < \mathbf{I}m\lambda < \pi/\varepsilon$.

(d) *There exists a* $\delta > 0$ *and a complex function* $\lambda(\varepsilon)$ *with continuous derivative* $\lambda'(\varepsilon)$ *defined in* $[\varepsilon_0 - \delta, \varepsilon_0 + \delta]$. *The function satisfies* $\Delta_\varepsilon(\lambda(\varepsilon)) = 0$ *for every* $[\varepsilon_0 - \delta, \varepsilon_0 + \delta]$, $\lambda(\varepsilon_0) = \pm i\sigma_0$, *and* $\mathbb{R}e\lambda'(\varepsilon_0) > 0$.

(ii) *For sufficiently small* ε *the equation* (3.3) *have exactly two simple real negative roots* $\lambda_1(\varepsilon)$ *and* $\lambda_2(\varepsilon)$ *such that*: $\lambda_1(\varepsilon) \nearrow 0$, $\lambda_2(\varepsilon) \searrow -a$, *as* $\varepsilon \to 0^+$. *The remaining complex roots verify the condition*:

$$\mathbb{R}e\lambda_n(\varepsilon) \to -\infty, \qquad \varepsilon \to 0^+, \qquad \forall n \geq 3.$$

(iii) *For each* $\varepsilon \in (0, \varepsilon_0)$ *there are precisely two real roots of Eq.* (3.4) $z_1(\varepsilon)$ *and* $z_2(\varepsilon)$ *such that* $z_1(\varepsilon) > 0$, $z_2(\varepsilon) < 0$ *and* $z_1(\varepsilon) \searrow 0$, $z_2(\varepsilon) \nearrow -a$ *as* $\varepsilon \to 0^+$. *The remaining complex roots have negative real parts.*

Proof. Item (i) is proved in [5], except (i)-(d).

Let us write $\Delta_\varepsilon(\lambda) = \Delta_\varepsilon(\mu + i\sigma)$ as

$$G(\mu, \sigma, \varepsilon) = \begin{bmatrix} \mu^2 - \sigma^2 + a\mu + b\varepsilon e^{-\mu}\cos\sigma \\ 2\mu\sigma + a\sigma - b\varepsilon e^{-\mu}\sin\sigma \end{bmatrix} = 0.$$

By (i)-(b), we know that $G(0, \sigma_0, \varepsilon_0) = 0$, where $\sigma_0 = \pm\xi/\varepsilon_0$, $a < b\varepsilon_0 < a\pi/2$. If the Jacobian of G with respect to μ and σ is different from zero at $(0, \sigma_0, \varepsilon_0)$, by the implicit function theorem there exists a $\delta > 0$ and two continuously differentiable functions $\mu(\varepsilon)$, $\sigma(\varepsilon)$ defined on $[\varepsilon_0 - \delta, \varepsilon_0 + \delta]$ such that

$$J(\mu, \sigma, \varepsilon) = \begin{bmatrix} 2\mu + a - b\varepsilon e^{-\mu}\cos\sigma & -2\sigma - b\varepsilon e^{-\mu}\sin\sigma \\ 2\sigma + b\varepsilon e^{-\mu}\sin\sigma & 2\mu + a - b\varepsilon e^{-\mu}\cos\sigma \end{bmatrix}$$

and

$$J(0, \sigma_0, \varepsilon_0) = \begin{bmatrix} A & -B \\ B & A \end{bmatrix},$$

where $A = a - b\varepsilon_0 \cos\sigma_0$, $B = 2\sigma_0 + b\varepsilon_0 \sin\sigma_0$. Thus, $\det J = A^2 + B^2 \neq 0$, since $B = \sigma_0(2 + a) \neq 0$, due to $a\sigma_0 = b\varepsilon_0 n \sin\sigma_0$. We know that

$$G(\mu(\varepsilon), \sigma(\varepsilon), \varepsilon) = 0, \qquad \forall \varepsilon \in [\varepsilon_0 - \delta, \varepsilon_0 + \delta],$$

thus $dG/d\varepsilon = 0$, or

$$\frac{\partial G}{\partial \mu}\frac{d\mu}{d\varepsilon} + \frac{\partial G}{\partial \sigma}\frac{d\sigma}{d\varepsilon} + \frac{\partial G}{\partial \varepsilon} = 0.$$

At the point $(0, \sigma_0, \varepsilon_0)$, the last equation has the form

$$\begin{aligned} A\mu' - B\sigma' &= -b\cos\sigma_0, \\ B\mu' + A\sigma' &= b\sin\sigma_0. \end{aligned}$$

After a long but straightforward computation, we get

$$\mu'(\varepsilon_0) = \frac{1}{A^2 + B^2}\left[b^2 + \frac{a}{\varepsilon_0^2}\sigma_0^2\right] > 0.$$

Let us prove (ii). It is clear that (3.3) has real roots iff the graphics of functions $g_1(x) = x^2 + ax$ and $g_2(x) = -\varepsilon b e^{-x}$ have crossing points. To this end, it is enough to choose ε sufficiently small. Without loss of generality, we can choose ε^* small enough so that $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$ verify the following:

$$-\frac{a}{2} < \lambda_1(\varepsilon) \leq 0, \qquad -a \leq \lambda_2(\varepsilon) < -\frac{a}{2}, \qquad \forall \varepsilon \in (0, \varepsilon^*].$$

From (3.3), we get that:

$$\frac{dx}{d\varepsilon} = \frac{-be^{-x}}{2x + a - \varepsilon b e^{-x}}.$$

Let x^* be the unique root of the equation $2x + a - \varepsilon b e^{-x} = 0$. It is easy to see that $-a/2 \leq x^* < 0$, $\forall \varepsilon \in (0, \varepsilon^*]$. Hence, $d\lambda_1(\varepsilon)/d\varepsilon < 0$, and $d\lambda_2(\varepsilon)/d\varepsilon > 0$, $\forall \varepsilon \in (0, \varepsilon^*]$. From the last inequalities certainly it follows that: $\lambda_1(\varepsilon) \nearrow 0$ and $\lambda_2(\varepsilon) \searrow -a$, as $\varepsilon \to 0^+$.

Let $\{\lambda_n(\varepsilon)\}_{n\geq 3}$ be the complex roots of Eq. (3.3). Suppose that $\mathbb{R}e\lambda_n(\varepsilon) \to -\infty$ as $\varepsilon \to 0^+$ for any $n \geq 3$ is not true. Then, there exists an $n_0 \geq 3$ and $M < 0$ such that

$$\mathbb{R}e\lambda_{n_0}(\varepsilon) \geq M, \qquad \forall \varepsilon \in (0, \varepsilon^*].$$

Let us set $\lambda_{n_0}(\varepsilon) = x(\varepsilon) + iy(\varepsilon)$, we claim that there exists a constant $\nu > 0$, independent on ε such that : $|y(\varepsilon)| \leq \nu$, $\forall\varepsilon \in [0, \varepsilon^*]$. Substituting $\lambda_{n_0}(\varepsilon) = x(\varepsilon) + iy(\varepsilon)$ in (3.3) and separating real and imaginary parts, we get:

$$\begin{aligned} x^2(\varepsilon) - y^2(\varepsilon) + ax(\varepsilon) + b\varepsilon e^{-x(\varepsilon)} \cos y(\varepsilon) &= 0 \\ 2x(\varepsilon)y(\varepsilon) + ay(\varepsilon) - b\varepsilon e^{-x(\varepsilon)} \sin y(\varepsilon) &= 0. \end{aligned} \tag{3.5}$$

From (3.5), it follows: $|y(\varepsilon)| \leq \nu$, with $\nu = \sqrt{M^2 + a\,|M| + b\varepsilon^* e^{-M}}$; which proves our assertion.

Let $\mathcal{D}$ be the subset of $\mathbf{C}$ defined as follows:

$$\mathcal{D} = \{z \in \mathbf{C} : z = x + iy,\ M \leq x \leq 0,\ |y| \leq \nu\}.$$

From the compactness of $\mathcal{D}$ and the fact $\lambda_{n_0}(\varepsilon) \in \mathcal{D}$, for any $\varepsilon \in (0, \varepsilon^*]$, we get $\lambda_{n_0}(\varepsilon) \to \lambda^* \in \mathcal{D}$, as $\varepsilon \to 0^+$.

We are going to prove that λ^* is a root of the equation $\Delta_0(\lambda) = \lambda^2 + a\lambda = 0$. Indeed,

$$\begin{aligned} |\Delta_0(\lambda^*)| &\leq |\Delta_0(\lambda^*) - \Delta_\varepsilon(\lambda^*)| + |\Delta_\varepsilon(\lambda^*) - \Delta_\varepsilon(\lambda_{\nu_0}(\varepsilon))| \\ &\leq |\Delta_0(\lambda^*) - \Delta_\varepsilon(\lambda^*)| + N|\lambda_{n_0}(\varepsilon) - \lambda^*| \to 0,\ \varepsilon \to 0^+, \end{aligned}$$

where

$$N = \sup\left\{\left|\frac{\partial\Delta_\varepsilon(\lambda)}{\partial\lambda}\right| : \lambda \in \mathcal{D},\ \varepsilon \in [0, \varepsilon^*]\right\}.$$

On the other hand, we know that: $\lim_{\varepsilon\to 0^+} \Delta_\varepsilon(\lambda) = \Delta_0(\lambda)$, uniformly on compact subset of $\mathbf{C}$. Hence, $\lambda^* = -a$ or $\lambda^* = 0$. If $\lambda^* = -a$, then λ^* should be a zero of $\Delta_0(\lambda)$ of multiplicity two, because $\lambda_2(\varepsilon) \to \lambda^*$ and $\lambda_{n_0}(\varepsilon) \to \lambda^*$ as $\varepsilon \to 0^+$; which is a contradiction. Analogously we treat the case when $\lambda^* = 0$.

The first part of the assertions in (iii) follows immediately from the facts that the graphics of the functions $f_1(x) = x^2 + ax$ and $f_2(x) = \varepsilon be^{-x}$ have two crossing points for any $\varepsilon > 0$, and $f_2(x) \to 0$, as $\varepsilon \to 0^+$, uniformly on compact subset of $\mathbb{R}$.

We shall show that Eq. (3.4) has not pure imaginary roots for $\varepsilon \in (0, \varepsilon_0)$. Indeed, $z = iy$ is a root of (3.4) iff

$$y^2 = -\varepsilon b \cos y, \qquad ay = -\varepsilon b \sin y. \tag{3.6}$$

From the second equation in (3.6), we get:

$$|y| \leq \varepsilon b/a.$$

Since $\varepsilon < \varepsilon_0$ and $\varepsilon_0 \in (a/b, a\pi/2b)$, it follows that $|y| < \pi/2$. But, in this case y does not verify the first equation in (3.6). This proves that all the complex roots of (3.4) have non-zero real parts, for $\varepsilon \in (0, \varepsilon_0)$.

Analogously to the proof of part (ii), we can show that:

$$\mathbb{R}e z_n(\varepsilon) \to -\infty, \qquad \text{as } \varepsilon \to 0^+, \quad n \geq 3.$$

Hence, $\mathbb{R}e z_n(\varepsilon) < 0$, $\forall n \geq 3$, $\varepsilon < \varepsilon_0$. If it is not the case, there exists a $z_{n_0}(\varepsilon)$ and $\varepsilon^* \in (0, \varepsilon)$ such that $\mathbb{R}e z_{n_0}(\varepsilon^*) > 0$. Then, there is an $\bar{\varepsilon} \in (0, \varepsilon_0)$ such that $\mathbb{R}e z_{n_0}(\bar{\varepsilon}) = 0$, which is absurd. ∎

LEMMA 3.1. *The global attractor S_ε of the RFDE* (2.3) *is homeomorphic to S^1 for any $\varepsilon \in (0, a/b]$.*

Proof. From the above made remarks, we know that $\{T_\varepsilon(\tau)\}_{\tau \geq 0}$ is a gradient system for $\varepsilon \in (0, a/b]$. Now, assertions (i)-(a) and (iii) in Proposition 3.1 imply the hyperbolicity of φ^* and φ^{**} for any $\varepsilon \in (0, \varepsilon_0)$. Moreover, dim $W^u(\varphi^*) = 0$ and dim $W^u(\varphi^{**}) = 1$ for any $\varepsilon \in (0, \varepsilon_0)$. Then we can conclude that $S_\varepsilon = W^u(\varphi^{**}) \cup \{\varphi^*\}$, $\forall \varepsilon \in (0, a/b]$. Since S_ε is connected, it follows that $W^u(\varphi^{**})$ is connecting the two fixed points. Finally, taking into account that $\{T_\varepsilon(\tau)\}_{\tau \geq 0}$ is gradient system for $\varepsilon \in (0, a/b]$ and completely continuous for $\tau \geq 1$, and $E = \{\varphi^*, \varphi^{**}\}$ is bounded, then $W^u(\varphi^{**})$ is one dimensional embedded submanifold of C° (see [2, Thm. 3.8.6.) Hence, S_ε must be homeomorphic to S^1 for any $\varepsilon \in (0, a/b]$. ∎

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